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# SOLUTION OF THE PROBLEM OF TORSION OF AN ELASTIC ROD OF $s$-GONAL CROSS SECTION USING THE BOUNDARY EXTENSION METHOD

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*The problem of torsion of an elastic rod with a regular $s$-gon in cross section is solved in analytical form using the boundary extension method. The method is based on the introduction of a wide rectangular region and special rapidly converging Fourier series.*

**Key words:** *elastic rod, torsion, boundary extension method.*

Saint-Venant [1] obtained an exact solution of the problem of torsion of a rod with a regular triangle in cross section by choosing polynomials from Cartesian coordinates. Timoshenko and Goodier [2] obtained an analytical solution for the torsion of a rod of rectangular cross section using the variable separation method. Arutyunyan and Abramyan [3] developed this approach by solving problems for cross sections of special shapes whose boundaries coincide with the coordinate lines of Cartesian, cylindrical or elliptic coordinate systems. Muskhelishvili proposed [4] a method of the theory of functions of a complex variable, but it is difficult to use because of the complexity of constructing conformal mappings.

The boundary expansion method proposed in the present work provides high-accuracy analytical solutions of elastic problems for bodies of complex shape. The problem of torsion of a rod whose cross section is a regular $s$-square is considered as one of the simplest examples. This problem is of applied significance because polygonal rods in torsion are often used in engineering.

We divide a regular $s$-square into $2s$ identical rectangular triangles with an acute angle $\alpha = \pi/s$. The coordinate origin is placed at the center of the polygon, as shown in Fig. 1. Then, we formulate the following problem for the stress function in the triangle $\Omega = \Delta OAB$:

$$\Delta U = -2, \qquad \frac{\partial U}{\partial y}\Big|_{y=0} = U\Big|_{x=a} = \frac{\partial U}{\partial x}\sin\alpha - \Big(\frac{\partial U}{\partial y}\Big|_{y=x\tan\alpha}\Big)\cos\alpha = 0,$$
$$U \in \{L_p^\alpha(\Omega), C^{(2)}(\Omega)\} \tag{1}$$

($L_p^\alpha$ are the classes of Sobolev–Liouville functions). To calculate the derivatives and then expand the unknown functions in Fourier series, it is necessary that in (1) the conditions of smoothness of ($C^{(2)}$) and integrability be satisfied [5]. As the region expanded with respect to the triangle $OAB$, we choose the rectangle

$$\Omega = OABC, \qquad 0 \leqslant x \leqslant a, \quad 0 \leqslant y \leqslant b$$

with the sides $OA = a$ and $AB = b = a\tan\alpha$. The boundary conditions on the sides of this rectangle are written as

$$\frac{\partial U}{\partial y}\Big|_{y=0} = U\Big|_{x=a} = 0, \qquad \frac{\partial U}{\partial y}\Big|_{y=b} = f_3(x), \qquad U\Big|_{x=0} = f_4(y).$$

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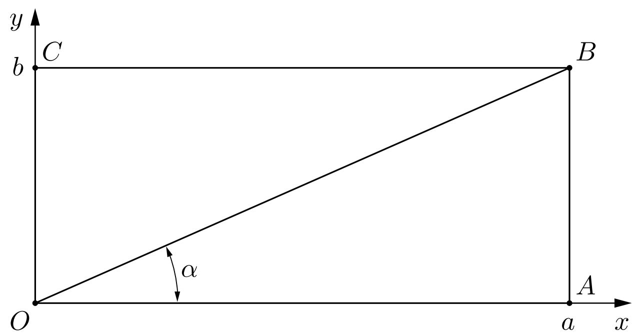


Fig. 1. Triangular region and the region expanded with respect to it.

On the two sides of the rectangle $OABC$ that coincide with the legs of the triangle at $y = 0$ and $x = a$, the boundary conditions coincide with the boundary conditions of the primary problem (1), and on the other two sides of the rectangle for $y = b$ and $x = 0$, the conditions $f_3(x)$ and $f_4(y)$ are unknown and are obtained from the condition on the hypotenuse $OB$ of the triangle of the primary problem — the last of the boundary conditions in (1). For this, we make the replacement

$$U(x, y) = M(x, y) + V(x, y), \qquad (M, V) \in L_p^\alpha(\Omega),$$
$$(M, V) \in \{C^{(4)}(0 < x < a),\ C^{(3)}(0 \leqslant x \leqslant a),\ C^{(3)}(0 < y < b),\ C^{(2)}(0 \leqslant y \leqslant b)\}, \tag{2}$$

where the boundary function $M$ on the sides of the rectangle should satisfy the same conditions as the function $U$:

$$\frac{\partial M}{\partial y}\Big|_{y=0} = M\Big|_{x=a} = 0, \qquad \frac{\partial M}{\partial y}\Big|_{y=b} = f_3(x), \qquad M\Big|_{x=0} = f_4(y), \qquad M \in C^{(2)}(\Omega). \tag{3}$$

In addition to conditions (3), we require that the second partial derivative $M_{xx}$ satisfy the additional boundary conditions

$$M_{xx}\Big|_{x=a} = U_{xx}\Big|_{x=a} = f_6(y), \qquad M_{xx}\Big|_{x=0} = U_{xx}\Big|_{x=0} = f_8(y). \tag{4}$$

The use of additional conditions (4) does not put any constraints on the solution of the problem, since the functions $f_6(y)$ and $f_8(y)$ are unknown, but these conditions allow one to use rapidly converging Fourier series and calculate the second partial derivatives $U_{xx}$ and $U_{yy}$ on all sides of the rectangle. Formulas (3) and (4) contain four unknown functions $f_3(x)$, $f_4(y)$, $f_6(y)$, and $f_8(y)$ which are expressed in terms of the function $U$ and its partial derivatives on the boundaries of the expanded region $\Omega$. From the smoothness conditions (2), it follows that the indicated functions cannot be arbitrary and, at the angles of the rectangle, they should satisfy the matching conditions

$$f_4'(0) = f_6'(0) = f_8'(0) = f_3(a) = 0,$$
$$f_3(0) = f_4'(b), \qquad f_6'(b) = f_3''(a), \qquad f_8'(b) = f_3''(0). \tag{5}$$

Using the boundary function $M$, it is possible to construct rapidly converging Fourier series to find $U$. The simplest form of the function $M(x, y)$ is given by

$$M = \frac{y^2}{2b}\Big[f_3(x) - \Big(1 - \frac{x}{a}\Big)f_3(0) - \Big(\frac{x^3}{6a} - \frac{ax}{6}\Big)f_3''(a) - \Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)f_3''(0)\Big]$$
$$+ \Big(1 - \frac{x}{a}\Big)f_4(y) + \Big(\frac{x^3}{6a} - \frac{ax}{6}\Big)f_6(y) + \Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)f_8(y). \tag{6}$$

A direct check shows that, if the matching conditions (5) are satisfied, the function $M$ from (6) satisfies boundary conditions (3) and (4).

Because the functions $f_6(y)$ and $f_8(y)$ in relation (5) are determined through the second partial derivatives, we sequentially set $x = 0$ and $x = a$ to calculate these functions in the Poisson equation from (1). As a result, we have

$$U_{xx}\Big|_{x=0} + U_{yy}\Big|_{x=0} = -2, \qquad U_{xx}\Big|_{x=a} + U_{yy}\Big|_{x=a} = -2. \tag{7}$$

Using relations (3) and (4), we write auxiliary expressions in the form

$$\begin{gathered} U_{xx}\Big|_{x=0} = f_8(y), \qquad U\Big|_{x=0} = f_4(y) \quad \Rightarrow \quad U_{yy}\Big|_{x=0} = f_4''(y), \\ U\Big|_{x=a} = 0 \quad \Rightarrow \quad U_{yy}\Big|_{x=a} = 0. \end{gathered} \tag{8}$$

From (7) and (8), we obtain

$$f_6(y) = -2, \qquad f_8(y) = -2 - f_4''(y). \tag{9}$$

In view of (9), the matching conditions (5) can be simplified to

$$f_3(a) = f_3''(a) = f_4'(0) = f_4'''(0) = 0, \qquad f_3(0) = f_4'(b), \qquad f_3''(0) = -f_4'''(b). \tag{10}$$

With the use of (9) or (10), the expression for the function $M$ from (6) are written as

$$\begin{aligned} M = \frac{y^2}{2b}\Big[f_3(x) - \Big(1 - \frac{x}{a}\Big)f_3(0) - \Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)f_3''(0)\Big] + (ax - x^2) \\ + \Big(1 - \frac{x}{a}\Big)f_4(y) - \Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)f_4''(y), \qquad x \in [0,a], \quad y \in [0,b]. \end{aligned} \tag{11}$$

For the function $V$ in the expanded region $\Omega$ for the inhomogeneous Poisson equation, we obtain the following boundary-value problem with homogeneous boundary conditions:

$$\Delta V = -2 - \Delta M, \qquad \frac{\partial V}{\partial y}\Big|_{y=0} = V\Big|_{x=a} = \frac{\partial V}{\partial y}\Big|_{y=b} = V\Big|_{x=0} = 0. \tag{12}$$

From the solution of the Euler–Lagrange equation, we find the eigenfunctions and eigenvalues $G_{m,n}$ and $\lambda_{m,n}$

$$\Delta G_{m,n} + \lambda_{m,n} G_{m,n} = 0, \qquad \frac{\partial G_{m,n}}{\partial y}\Big|_{y=0} = G_{m,n}\Big|_{x=a} = \frac{\partial G_{m,n}}{\partial y}\Big|_{y=b} = G_{m,n}\Big|_{x=0} = 0.$$

For the rectangle, the eigenfunctions and eigenvalues have the form

$$G_{m,n} = \sin\frac{m\pi x}{a}\,\cos\frac{n\pi y}{b}, \qquad \lambda_{m,n} = \Big(\frac{m\pi}{a}\Big)^2 + \Big(\frac{n\pi}{b}\Big)^2,$$

$$m = 1, 2, \ldots, \quad n = 0, 1, \ldots.$$

The required functions $V(x,y)$, $f_3(x)$, and $f_4(y)$, each in its own range of definition are expanded in rapidly converging Fourier series:

$$\begin{gathered} U = M + V, \qquad V = \sum_{\substack{m=1\\n=1}}^{\infty} v_{m,n}\sin\frac{m\pi x}{a}\,\cos\frac{n\pi y}{b} + \sum_{m=1}^{\infty} v_{m,0}\sin\frac{m\pi x}{a}, \\ f_3(x) = \Big(1 - \frac{x}{a}\Big)f_3(0) + \Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)f_3''(0) + \sum_{m=1}^{\infty} f_{3,m}\sin\frac{m\pi x}{a}, \\ f_4(y) = f_4^* + \frac{y^2}{2b}\,f_3(0) + f_4'''(b)\Big(\frac{y^4}{24b} - \frac{by^2}{12}\Big) + \sum_{n=1}^{\infty} f_{4,n}\cos\frac{n\pi y}{b}. \end{gathered} \tag{13}$$

In the Fourier series (13), the terms ahead of the sums for $f_3(x)$ and $f_4(y)$ (boundary functions) are chosen so that theses series converge uniformly in the corresponding regions of their definition, together with the derivatives up to the fourth order with respect to the variable $x$ and derivatives up to the fifth order with respect to the variable $y$ inclusive [6]. The Fourier series for $U(x)$ converges uniformly in $\Omega$, together with the fourth-order partial derivatives with respect to the variable $x$ and derivatives up to the third order with respect to the variable $y$ inclusive. This leads to the following estimate of the series coefficients:

$$v_{m,n} \sim \frac{1}{m^5 n^4} \quad (n \neq 0), \qquad (v_{m,0}, f_{3,m}) \sim \frac{1}{m^5}, \qquad f_{4,n} \sim \frac{1}{n^6}. \tag{14}$$

From (14) it follows that the torsion function $U$ and its partial derivatives $U_x$ and $U_y$, in terms of which the shear stress in the elastic rod are expressed, are calculated with the error

$$\delta U \sim \max\left((m_0+1)^{-5}, (n_0+1)^{-4}\right) = \delta_0, \qquad \delta U_x \sim \max\left((m_0+1)^{-4}, (n_0+1)^{-4}\right),$$
$$\delta U_y \sim \max\left((m_0+1)^{-5}, (n_0+1)^{-3}\right). \tag{15}$$

The validity of calculating the partial derivatives $U_x$ and $U_y$ of the Fourier series used to determine the solution $U$ is due to the fact that the boundary function $M$ in the form of (11) and the boundary functions from (13) in the representation of $f_3(x)$ and $f_4(y)$ satisfy the conditions of the theorem on term-by-term differentiation of Fourier series [6]. Estimates (14) and (15) imply that, in the Fourier series (13) for $U$, one should take into account only those coefficients from the set $\{v_{m,n}\}$ whose numbers $m$ and $n$ satisfy the inequality $(m+1)^5(n+1)^4 \leqslant \delta_0^{-1}$, where $n \geqslant 1$. In the construction of the solution of the problem to within $10^{-4}$, it is sufficient to take four term ($n_0 = m_0 = 4$) in each of the sums over the indices $n$ and $m$. Omitting small coefficients of the higher order, we obtain the following set of unknowns:

$$v_{m,0},\ v_{m,1},\ v_{1,2},\ v_{1,3},\ f_3(0),\ f_3''(0),\ f_{3,m},\ f_4^*,\ f_4'''(b),\ f_{4,n},$$
$$m = 1, \ldots, 4, \qquad n = 1, \ldots, 4,$$

Substituting $U$ from (13) and $M$ from (11) into the Poisson equation (1), we have

$$\sum_{\substack{m=1\\n=0}}^{\infty} v_{m,n}\lambda_{m,n}G_{m,n} = \frac{y^2}{2b}\Big[f_3''(x) - \Big(1 - \frac{x}{a}\Big)f_3''(0)\Big] - \Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)f_4^{\mathrm{IV}}(y)$$
$$+ \frac{1}{b}\Big[f_3(x) - \Big(1 - \frac{x}{a}\Big)f_3(0) - \Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)f_3''(0)\Big]. \tag{16}$$

Replacement of the expression $f_3(x)$ by relation (13) simplifies equality (16):

$$\sum_{\substack{m=1\\n=1}}^{\infty} v_{m,n}\lambda_{m,n}G_{m,n} + \sum_{m=1}^{\infty} v_{m,0}\,\frac{m^2\pi^2}{a^2}\sin\frac{m\pi x}{a} = -\frac{y^2}{2b}\sum_{m=1}^{\infty} f_{3,m}\,\frac{m^2\pi^2}{a^2}\sin\frac{m\pi x}{a}$$
$$+ \frac{1}{b}\sum_{m=1}^{\infty} f_{3,m}\sin\frac{m\pi x}{a} - \Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)\Big[\frac{1}{b}f_4'''(b) + \sum_{n=1}^{\infty} f_{4,n}\,\frac{n^4\pi^4}{b^4}\cos\frac{n\pi y}{b}\Big]. \tag{17}$$

Multiplying Eq. (17) by $G_{m,n}$ and integrating the result over the rectangular region $\Omega$, we find the coefficients $v_{m,0}$ and $v_{m,n}$ in explicit form

$$v_{m,0} = -f_{3,m}\,\frac{b}{6} + f_{3,m}\,\frac{a^2}{m^2\pi^2 b} + f_4'''(b)\,\frac{2a^4}{m^5\pi^5 b},$$
$$v_{m,n} = \frac{2}{\lambda_{m,n}}\Big[-f_{3,m}(-1)^n\,\frac{m^2 b}{a^2 n^2} + f_{4,n}\,\frac{a^2 n^4 \pi}{m^3 b^4}\Big]. \tag{18}$$

To find the coefficients $f_3(0)$, $f_4^*$, $f_4'''(b)$, $f_{3,m}$, and $f_{4,n}$ we substitute the functions $M$ from (11) and $V$ from (13) into relation (3), and then the function $U$ into the last boundary condition from (1) specified on the hypotenuse $OB$ for $y = x\tan\alpha$. As a result, we have

$$\frac{b^2x^2}{2a^2}\sum_{m=1}^{\infty} f_{3,m}\,\frac{m\pi}{a}\cos\frac{m\pi x}{a} + b(a-2x) + \sum_{\substack{m=1\\n=1}}^{\infty} v_{m,n}\,\frac{an\pi}{b}\,\sin\frac{m\pi x}{a}\sin\frac{n\pi x}{a}$$

$$-\frac{b}{a}\Big[f_4^* + f_3(0)\,\frac{bx^2}{2a^2} + f_4'''(b)\,\frac{b^3}{12a^2}\Big(\frac{x^4}{2a^2} - x^2\Big) + \sum_{n=1}^{\infty} f_{4,n}\,\cos\frac{n\pi x}{a}\Big]$$

$$-\Big(x - \frac{x^2}{2a} - \frac{a}{3}\Big)\Big[f_3(0) + f_4'''(b)\,\frac{b^2}{6}\Big(\frac{3x^2}{a^2} - 1\Big) - \sum_{n=1}^{\infty} f_{4,n}\,\frac{n^2\pi^2}{b}\,\cos\frac{n\pi x}{a}\Big]$$

$$+\sum_{\substack{m=1\\n=1}}^{\infty} v_{m,n}\,\frac{bm\pi}{a}\cos\frac{m\pi x}{a}\cos\frac{n\pi x}{a} + \sum_{m=1}^{\infty} v_{m,0}\,\frac{bm\pi}{a}\,\cos\frac{m\pi x}{a}$$

$$-x\sum_{m=1}^{\infty} f_{3,m}\sin\frac{m\pi x}{a} - (a-x)\Big[f_3(0)\,\frac{x}{a} + f_4'''(b)\,\frac{b^2}{6}\Big(\frac{x^3}{a^3} - \frac{x}{a}\Big) - \sum_{n=1}^{\infty} f_{4,n}\,\frac{n\pi}{b}\sin\frac{n\pi x}{a}\Big]$$

$$+\Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)\Big[f_4'''(b)x + \sum_{n=1}^{\infty} f_{4,n}\,\frac{an^3\pi^3}{b^3}\sin\frac{n\pi x}{a}\Big].$$

Eliminating $v_{m,0}$ and $v_{m,n}$ by means of their expressions from (18), we obtain

$$-f_3(0)\Big(2x - \frac{3x^2}{2a} - \frac{a}{3} + \frac{b^2x^2}{2a^3}\Big) - f_4^*\,\frac{b}{a}$$

$$+f_4'''(b)\Big[\frac{x^3}{2} - \frac{x^4}{6a} - \frac{ax^2}{3} - \frac{b^4x^4}{24a^5} + \frac{b^4x^2}{12a^3} - \frac{b^2}{6}\Big(\frac{3x^2}{a^2} - 1\Big)\Big(x - \frac{x^2}{2a} - \frac{a}{3}\Big)$$

$$-\frac{b^2}{6}\Big(\frac{x^3}{a^2} - x\Big)\Big(1 - \frac{x}{a}\Big) + 2a^3\Big(\frac{1}{90} - \frac{x^2}{12a^2} + \frac{x^3}{12a^3} - \frac{x^4}{48a^4}\Big)\Big]$$

$$+\sum_{m=1}^{m_0} f_{3,m}\Big\{\Big[\frac{b^2x^2m\pi}{2a^3} - \frac{2m^3b^2\pi}{a^3}\sum_{p=1}^{20}(-1)^p\,\frac{1}{\lambda_{m,p}p^2}\,\cos\frac{p\pi x}{a} - \frac{b^2m\pi}{6a} + \frac{a}{m\pi}\Big]\cos\frac{m\pi x}{a}$$

$$-\Big[x + \frac{2\pi m^2}{a}\sum_{p=1}^{20}(-1)^p\frac{1}{\lambda_{m,p}p}\,\sin\frac{p\pi x}{a}\Big]\sin\frac{m\pi x}{a}\Big\}$$

$$+\sum_{n=1}^{m_0} f_{4,n}\Big\{\Big[\frac{n^2\pi^2}{b}\Big(x - \frac{x^2}{2a} - \frac{a}{3}\Big) - \frac{b}{a}\cos\frac{n\pi x}{a} + \frac{2an^4\pi^2}{b^3}\sum_{q=1}^{20}\frac{1}{\lambda_{q,n}q^2}\cos\frac{q\pi x}{a}\Big]\cos\frac{n\pi x}{a}$$

$$+\Big[\frac{an\pi}{b}\Big(1 - \frac{x}{a}\Big) + \frac{an^3\pi^3}{b^3}\Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big) + \frac{2a^3n^5\pi^2}{b^5}\sum_{q=1}^{20}\frac{1}{\lambda_{q,n}q^3}\,\sin\frac{q\pi x}{a}\Big]\sin\frac{n\pi x}{a}\Big\} = b(2x-a). \tag{19}$$

The left and right sides of Eq. (19) are expressed by various functions of the variable $x \in [0,a]$, but with an appropriate choice of the coefficients $f_3(0)$, $f_4^*$, $f_4'''(b)$, $f_{3,m}$, $f_{4,n}$ $\forall\, x \in [0,a]$, the values of these sides are identical. Sequentially setting $x = 0$ and $x = a$ in Eq. (19) and integrating its left and right sides with respect to $x$ in the range of $[0,a]$, we obtain three algebraic equations. Multiplying the left and right sides of (19) by $\cos(q\pi x/a)$ and then by $\sin(q\pi x/a)$ $(q = 1, 2, \ldots)$ and integrating over the interval $[0,a]$, we arrive at a system closed for $f_3(0)$, $f_4^*$, $f_4'''(b)$, $f_{3,m}$, and $f_{4,n}$ which can also be obtained in finite explicit form. (This system is not given here because of space limitations.) In this case, because rapidly converging Fourier series are used, it is possible to employ a simpler pointwise method that satisfies boundary condition (19). For this, we confine each sum of Eq. (19)to a finite number of terms, setting $m = 1, \ldots, m_0$, $n = 1, \ldots, n_0$. Requiring that equality (19) be satisfied only at the calculated points on the hypotenuse of the triangle:

$$x = x_k = ak/(m_0 + n_0 + 2), \qquad k = 0, \ldots, m_0 + n_0 + 2, \tag{20}$$

from (19) we obtain a linear system of algebraic equations which are closed for $f_3(0)$, $f_4^*$, $f_4'''(b)$, $f_{3,m}$, and $f_{4,n}$. Solving this system, we find the values of $v_{m,n}$, $f_3(0)$, $f_4^*$, $f_{3,m}$, and $f_{4,n}$ and substitute them into expressions (13) for $V$, $f_3$, and $f_4$ and then into (6) for $M$ and in (2) to obtain solutions of the problem $U$. It should be noted that, in the case of using classical Fourier series, the pointwise method of calculating the coefficients is inapplicable. Because the function $U$ is represented in analytical form, the stress at any point of the rod cross section is calculated by the formulas

$$\sigma_{xz} = G_0\theta\Big\{\frac{y}{b}\sum_{m=1}^{\infty} f_{3,m}\sin\frac{m\pi x}{a} - \sum_{\substack{m=1\\n=1}}^{\infty} v_{m,n}\frac{\pi n}{b}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$-\Big(\frac{x^2}{2} - \frac{x^3}{6a} - \frac{ax}{3}\Big)\Big(f_4'''(b)\,\frac{y}{b} - \sum_{n=1}^{\infty} f_{4,n}\,\frac{\pi^3 n^3}{b^3}\,\sin\frac{n\pi y}{b}\Big)$$

$$+\Big(1 - \frac{x}{a}\Big)\Big[\frac{y}{b}\,f_3(0) + f_4'''(b)\Big(\frac{y^3}{6b} - \frac{by}{6}\Big) - \sum_{n=1}^{\infty} f_{4,n}\,\frac{\pi n}{b}\sin\frac{n\pi y}{b}\Big]\Big\},$$

$$\sigma_{yz} = -G_0\theta\Big\{\frac{\pi y^2}{2ab}\sum_{m=1}^{\infty} f_{3,m} m\cos\frac{m\pi x}{a} + \sum_{m=1}^{\infty} v_{m,0}\,\frac{m\pi}{a}\,\cos\frac{m\pi x}{a}$$

$$+\frac{\pi}{a}\sum_{\substack{m=1\\n=0}}^{\infty} m v_{m,n}\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b} + a - 2x$$

$$-\frac{1}{a}\Big[f_4^* + \frac{y^2}{2b}f_3(0) + f_4'''(b)\Big(\frac{y^4}{24b} - \frac{by^2}{12}\Big) + \sum_{n=0}^{\infty} f_{4,n}\cos\frac{n\pi y}{b}\Big]$$

$$-\Big(x - \frac{x^2}{2a} - \frac{a}{3}\Big)\Big[\frac{1}{b}\,f_3(0) + f_4'''(b)\Big(\frac{y^2}{2b} - \frac{b}{6}\Big) - \sum_{n=1}^{\infty} f_{4,n}\,\frac{\pi^2 n^2}{b^2}\,\cos\frac{n\pi y}{b}\Big]\Big\}.$$

Here $G_0$ is Young's elastic modulus, $\theta$ is the angle of torsion per unit length of the bar, $v_{m,n}$ and $v_{m,0}$ should be taken from (18). For $m_0 = n_0 = 4$, $a = 1$, and $b = \sqrt{3}$ for an elastic rod of triangular cross section, from the linear algebraic system we find the following values of the coefficients $f_3(0)$, $f_4^*$, $f_4'''(b)$, $f_{3,m}$, $f_{4,n}$ (only four decimal places are given): $f_3(0) = -1.7320$, $f_4^* = 0.6666$, $f_4'''(b) = 6.354 \cdot 10^{-6}$, $f_{3,1} = -1.164 \cdot 10^{-6}$, $f_{3,2} = 1.020 \cdot 10^{-7}$, $f_{3,3} = 9.465 \cdot 10^{-8}$, $f_{3,4} = 9.285 \cdot 10^{-8}$, $f_{4,1} = -1.727 \cdot 10^{-7}$, $f_{4,2} = -3.086 \cdot 10^{-9}$, $f_{4,3} = -6.891 \cdot 10^{-10}$, and $f_{4,4} = -6.949 \cdot 10^{-11}$; $U(0, 0) = 0.6666$. The error of the approximate solution has an order of magnitude smaller than $10^{-5}$ and is determined as the difference between the values of the stress function at the center of the elastic rod $U(0, 0)$ obtained in the approximate solution in the form of (13) and the exact solution [1]. Setting $a = 1$ and $b = 1/\sqrt{3}$ in Eq. (19), we find the solution of the problem of torsion of an elastic rod of hexagonal cross section. We obtain the following values of the coefficients $f_3(0)$, $f_4^*$, $f_4'''(b)$, $f_{3,m}$, and $f_{4,n}$: $f_3(0) = -1.4358$, $f_4^* = -0.0423$, $f_4'''(b) = -92.5468$, $f_{3,1} = 5.4142$, $f_{3,2} = 1.7332$, $f_{3,3} = 0.7581$, $f_{3,4} = 0.1987$, $f_{4,1} = 0.3393$, $f_{4,2} = -0.0469$, $f_{4,3} = 0.0091$, and $f_{4,4} = 0.0022$; $U(0, 0) = 0.2613$.

To calculate the coefficients $f_3(0)$, $f_4^*$, $f_4'''(b)$, $f_{3,m}$, and $f_{4,n}$ and solve the problem for $m_0 = n_0 = 4$, we use system of 11 linear algebraic equations obtained from (19) at the points $x = x_k$ of a uniform grid (20). To verify whether the pointwise method satisfies boundary condition (19), we performed the following numerical experiment.

We substitute the above coefficients $f_3(0)$, $f_4^*$, $f_4'''(b)$, $f_{3,m}$, and $f_{4,n}$ into (19) and denote the difference between the left and right sides by $\delta Y(x)$. By the construction, the equality $\delta Y(x_k) = 0$ should be satisfied at the calculated points $x = x_k$ and inequalities $\delta Y(x) \neq 0$ are satisfied at the intermediate points $x_k < x < x_{k+1}$. It was checked numerically that $\max |\delta Y(x)| < 10^{-6}\ \forall\ x \in [0, 1]$. This implies that boundary condition (19) is satisfied with high accuracy everywhere.

From this example, it follows that the proposed boundary expansion method allows one to obtain approximate solutions in analytical form with high accuracy with insignificant computational efforts. This method can also be used to solve problems for regions of more complex curvilinear shapes and nonlinear and dynamic two- and three-dimensional problems with moving boundaries. In the case of nonlinear problems, the system of algebraic equations is nonlinear, and for dynamic problems, we have a system of ordinary differential equations for time $t$; i.e., the expansion method is fairly universal and can be useful for applications.

Thus, the proposed boundary expansion method offers significant advantages over finite-difference methods.

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